May vortices produce a mass gap in $2D$ spin models at weak coupling *

Oleg Borisenko† and Peter Skala‡
Institut für Kernphysik, Technische Universität Wien
A–1040 Vienna, Austria

February 7, 2008

Abstract

We consider the $2D$ $SU(N)$ principal chiral model and discuss a vortex condensation mechanism which could explain the existence of a non-zero mass gap at arbitrarily small values of the coupling constant. The mechanism is an analogue of the vortex condensation mechanism of confinement in $4D$ non-Abelian gauge theories. We formulate a sufficient condition for the mass gap to be non-vanishing in terms of the behaviour of the vortex free energy. The $SU(2)$ model is studied in detail. In one dimension we calculate the vortex free energy exactly. An effective model for the center variables of the spin configurations of the $2D$ $SU(2)$ model is proposed and the $Z(2)$ correlation function is derived in this model. We define a $Z(2)$ mass gap in both the full and effective model and argue that they should coincide whenever the genuine mass gap is non-zero. We show via Monte-Carlo simulations of the $SU(2)$ model that the $Z(2)$ mass gap reproduces the full mass gap with perfect accuracy. We also test this mechanism in the positive link model which is an analogue of the positive plaquette model in gauge theories and find excellent agreement between the full and the $Z(2)$ mass gap.

*Work supported by Grant of Austrian Ministry of Science, Research and Arts BM:WFK GZ 45.403/3-IV/3a/
†email: oleg@ap3.bitp.kiev.ua
‡email: skala@kph.tuwien.ac.at

1 Introduction: Motivation and Problems

The nature of a non-zero string tension (ST) at any value of the bare coupling constant is one of the most striking puzzles of QCD. An analogous question exists in two-dimensional ($2D$) non-Abelian spin models with continuous symmetry group where the mass gap (MG) is expected to be non-zero at any temperature. Despite huge efforts in both cases, a solution of this problem has not yet been found. A promising conjecture which could explain the existence of a non-zero ST in QCD and thus permanent confinement of quarks is based on a vortex condensation mechanism. In this article we adjust the lattice formulation of this mechanism [1] to the case of $2D$ non-Abelian spin models. Though this paper is in a spirit of [2], our treatment of the problem differs in some aspects. We propose a simple effective model for the $Z(2)$ degrees of freedom in the $SU(2)$ principal chiral model which explains how this mechanism could work in the weak coupling region of the lattice model.

The idea that condensation of vortices may be responsible for confinement of static quarks in non-Abelian gauge theories with non-trivial center appeared already in the late seventies [3, 4]. The essential concept was taken from $Z(N)$ lattice gauge theory (LGT) where the Wilson loop was known to obey an area law in the strong coupling region. Since $Z(N)$ forms the center of the $SU(N)$ group it was suggested that $Z(N)$ vortices can be also present in the more complicated $SU(N)$ theory and play an essential role in generating a non-zero ST. However, $Z(N)$ LGT undergoes a phase transition at weak coupling to the deconfinement phase with the Wilson loop obeying a perimeter law. Since $Z(N)$ vortices in $Z(N)$ LGT may have a thickness of only one lattice spacing its contribution to the free energy becomes negligible in the weak coupling region where the system is well ordered at small distances. What concerns $SU(N)$ LGT, attempts to calculate an effective $Z(N)$ theory at small coupling making a perturbative expansion around $Z(N)$ solutions of the Yang-Mills equations only lowered the critical coupling but did not remove it to zero as is expected to be the case for the correct confinement mechanism [4].

In theories with continuous symmetry group like $SU(N)$, vortices may have, however, a thickness of not only one but many lattice spacings. Hence, there is the possibility to generalize the naive mechanism of $Z(N)$ models and include all the possible vortex configurations present in $SU(N)$ theories. Such a theory of confinement was developed in [1] where also a theorem was proved which makes a link between the behaviour of the Wilson loop and the vortex condensate. In this theory the definition of a vortex is actually not important. The only important issue is a change of vorticity, and the vortex condensate is defined as the free energy of such a change introduced by special singular $Z(N)$ gauge transformations. Over large distances typical configurations look like $Z(N)$ vortices, i.e. the basic field variables separated by such a distance are rotated by a $Z(N)$ element relatively to each other. The vortex condensation mechanism is manifestly gauge invariant and presumably gives a nice explanation of the coexistence of confinement at large scales and perturbative behaviour at short range. The mechanism was investigated in many papers and some results supporting its validity were found [5]-[12]:

- A strong coupling expansion of the vortex free energy up to the 12-th order demonstrates that vortex configurations produce a ST which coincides with the full ST up to this order [5]. This result is gauge independent.
- The $Z(N)$ Wilson loop was shown to carry all the ST to all orders of the strong coupling expansion [6], at least in the electric gauge. It was argued that $Z(N)$ Wilson loops in different gauges differ only by perimeter contributions.

- If the magnetic-flux free energy vanishes in the limit of a large uniform dilatation of a torus, the vortex free energy always decreases exponentially. It is sufficient to produce confinement. Using this property one can rigorously prove the lower confining bound in three-dimensional $U(N)$ LGT [7].

- Simple intuitive ideas as well as some analytical results show that a vortex mechanism is likely to lead to confinement at weak couplings also in the positive plaquette model and the Mack-Petkova model, which eliminates certain $Z(N)$ magnetic monopoles in $SU(N)$ LGT [8].

- The technique to evaluate the contribution of vortices of arbitrary thickness to the expectation value of any observable was developed in [9]. Such a "preaveraged" Wilson loop, i.e. calculated solely on the vortex contributions, exhibits confining behaviour. While not rigorous, this result potentially refers also to the weak coupling limit of the $3D$ $SU(2)$ model.

- It has been proved that even the classical, though not naive limit of $SU(N)$ LGT includes bare vortices in the continuum Lagrangian. They are labeled by the nontrivial center elements of $SU(N)$ and are supported on closed $2D$ surfaces in four dimensions [10]. A one-loop expansion in a particular background of such vortices shows instability of the vacuum implying that vortices must condense (become "fatter") in the quantum theory already at two loop order.

- Recently, the vortex condensation mechanism of confinement was studied in the so-called maximal center gauge of $SU(2)$ LGT [11]. The authors of these papers claimed that the Wilson loop computed with the center projected $Z(2)$ gauge field degrees of freedom carries almost the whole asymptotic ST. Consequently, excluding all $Z(2)$ vortices identified after projection leads to a vanishing ST.

- The $SU(2)$ partition function can be rewritten in the form of coupled $SU(2)/Z(2)$ and $Z(2)$ models which allows to give a proper interpretation of different $Z(2)$ excitations in the original model [12]. Using plausible assumptions one can establish a link between these excitations and the behaviour of the sign of the trace of the Wilson loop. It was shown via Monte-Carlo (MC) simulations that the sign of the trace of the Wilson loop carries all the information about the asymptotic behaviour of the fundamental string tension.

It should be stressed that earlier papers on the vortex condensation mechanism of confinement in continuum QCD cannot be regarded as an explanation of confinement [13]: In absence of a non-perturbative definition the introduction of vortex configurations into the QCD Lagrangian seems to be a completely ad-hoc procedure. Thus such models are not able to give an explanation in terms of dynamical reasons why vortices should become fat, i.e. why they are condensed.

In general, there are two ways of looking at the vortex condensation mechanism in lattice QCD. The first one results from the desire to find an analogy with the continuum theory and interprets the vortices responsible for confinement as an analogue of the Nielsen-Olesen vortices and the QCD vacuum as the so-called spaghetti vacuum. The second approach is based on the similarities between lattice QCD and $2D$ non-Abelian spin models where, while not so close to the continuum, one can give precise mathematical definitions to all quantities involved. Following

the latter idea, Mack and Petkova [1] formulated a condition which could be called confinement mechanism by a vortex condensate. We would like to emphasize that while the relation of this approach to the spaghetti vacuum is rather vague at the present stage of affairs the connection to $2D$ spin models is straightforward since one has precise definitions in both cases. Moreover, a quantity like the vortex condensate is expected to be a genuine non-perturbative quantity and thus it should be clear that it can be given a precise meaning only in a non-perturbative approach such as the formulation of QCD on a lattice.

In this article we consider the vortex mechanism in some details on the example of $2D$ spin models. The reason to deal with these models is the following: The nature of the MG is unknown despite the claim in the literature that its exact physical value is known. A vortex condensation mechanism is one possible candidate to explain the phenomenon of a non-zero MG. Moreover, this mechanism has an analogue in gauge theories. In view of all similarities between $2D$ spin models and $4D$ gauge theories we think it is useful and instructive to study this mechanism on the example of simpler $2D$ models.

This paper is organized as follows. As a first point we give a definition of the vortex free energy in terms of the original spin configurations. We introduce a vortex container which has the topology of a ring in $2D$ and which is specified by certain boundary conditions (BC). We prove a sufficient condition for the correlation function to decrease exponentially which is precisely the analogue of the Mack-Petkova theorem in LGT. This is done is Section 2.

Having identified the exponential decay of the vortex free energy as a sufficient condition for producing a non-vanishing MG the question arises what are the configurations of the spin field responsible for this rapid change of vorticity. Following the ideas of [6, 14] one can introduce a $Z(N)$ correlation function which measures the effect of vortices on the full correlation function. In spin models with global symmetry this is conceptually easier since we do not have to fix a gauge to uniquely define such a quantity. One can give the corresponding arguments [14] that this $Z(N)$ correlation function defines completely the large distance behaviour of the full correlation if a vortex mechanism is responsible for the non-vanishing MG. We shall construct such a correlation and derive an effective Ising-like model for $Z(2)$ excitations in Section 3. Furthermore, we calculate the correlation function numerically for the case of the $SU(2)$ principal chiral model using MC simulations. We find that the MG extracted from the $Z(2)$ correlation function agrees almost completely with the asymptotic MG of the $SU(2)$ model. A more delicate question is what one can expect in the case of the positive link (PL) model where all the thin vortices are eliminated. We introduce this model in analogy to the PP model [8] and study it in the $SU(2)$ case. In particular, we define and calculate $Z(2)$ correlations in the same way as in the full $SU(2)$ model. All these questions are subject of Section 4.

Our conclusions and discussion are presented in Section 5.

2 Vortex mechanism in spin models

In this section we give a precise formulation of the vortex condensation mechanism in two-dimensional spin models. A first hint that such a mechanism could be crucial for the existence of a non-zero MG came from the famous paper by Dobrushin and Shlosman [15]. These authors pointed out that the Mermin-Wagner theorem on the absence of spontaneous magnetization in two-dimensional spin systems follows from the intuitive idea that in such systems long Peierls contours cost only little free energy by making them thick. This is possible because the spins

can rotate slowly due to the continuous nature of the symmetry group.

In the following we derive a sufficient condition for the MG in two-dimensional $SU(N)$ spin models to be non-vanishing. Our derivation follows closely the one of the corresponding condition in LGT by Mack and Petkova [1]. Nevertheless, we choose to adduce it here because of pedagogical reasons and to make this paper self-contained.

For definiteness we consider the $SU(N)$ principal chiral model on a two-dimensional periodic lattice Λ^D. Its partition function is given by

$$Z(\beta) = \int \prod_{x \in \Lambda^D} D\mu(U_x) \exp[\beta \sum_l \mathrm{ReTr}(U_x U_{x+n}^\dagger)], \qquad U_x \in SU(N). \tag{1}$$

β is the coupling constant, $D\mu$ denotes the normalized Haar measure on $SU(N)$ and the integration extends over all sites x of the lattice. The sum in the exponent is a sum over all links $l \equiv (x,n)$ of Λ^D. According to the conventional scenario, the fundamental correlation function of spins separated by a distance R

$$\Gamma_R(\beta) = \langle \, \mathrm{ReTr}(U_0 U_R^\dagger) \, \rangle \tag{2}$$

is expected to decrease exponentially at any value of the coupling constant β if R is sufficiently large

$$\Gamma_R(\beta) \sim \exp(-m_c(\beta) R), \tag{3}$$

with $m_c(\beta)$ the MG.

Let us consider now sets of links on Λ^D which form closed loops on the dual lattice as shown in fig. 1. Following [1] we call the region of the original lattice enclosed by two such loops a vortex container T which has the topology of a ring in two dimensions. We arrange many such containers between the lattice sites 0 and R. Different containers may touch but not intersect each other. Let ∂T_i be the boundary (the set of circles $\bullet$ and squares $\blacksquare$ in fig. 1) and Λ_i the interior (the set of crosses $+$ in fig. 1) of the i^{th} container T_i. We define the complement Λ_c of Λ^D as $\Lambda_c = \Lambda^D / \prod_i \Lambda_i$. To make the following derivation more transparent we rename the $SU(N)$ variables U_x. Spins belonging to one of the vortex containers T_i are renamed U'_x, spins lying in the complement Λ_c $\bar{U}_x$.

We start by rewriting the path integral expression for the fundamental correlation function (2) in terms of the variables U'_x and $\bar{U}_x$

$$\begin{aligned} \Gamma_R(\beta) \quad &= \quad \frac{1}{Z} \int \prod_{x \in \Lambda_c} D\mu(\bar{U}_x) \, \mathrm{ReTr}(\bar{U}_0 \bar{U}_R^\dagger) \, \exp\left[\beta \sum_{l \in \Lambda_c} \mathrm{ReTr}(\bar{U}_x \bar{U}_{x+n}^\dagger) \right] \\ &\cdot \quad \prod_i \left\{ \int \prod_{x \in T_i} D\mu(U'_x) \, \exp[\beta {\sum_{l \in T_i}}' \mathrm{ReTr}(U'_x U'^\dagger_{x+n})] \prod_{x \in \partial T_i} \delta(U'_x \bar{U}_x^{-1}) \right\}. \end{aligned} \tag{4}$$

The group δ-function is necessary to avoid double integration over spins defined on the boundary ∂T_i of one of the containers. The product $\prod_i$ runs over all containers arranged between the lattice sites 0 and R. In the sum $\sum'$ in the exponent of the inner integral, links in the boundary of the

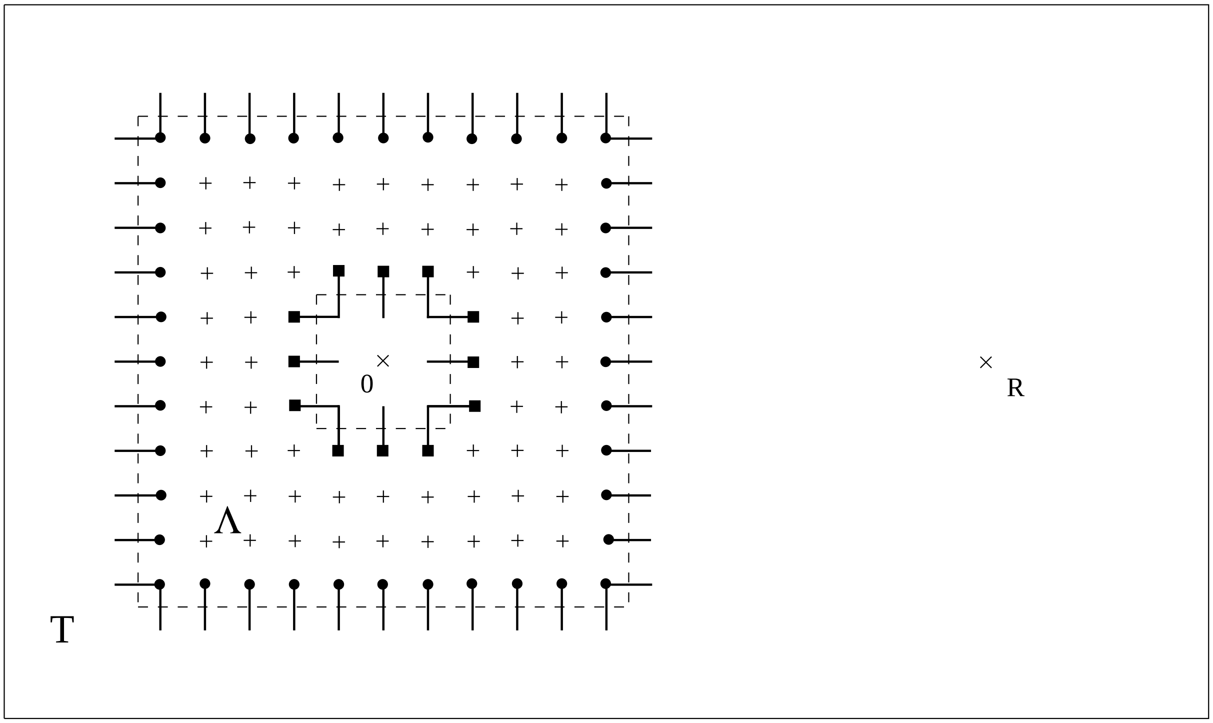

Figure 1: An example of a vortex container T on a two-dimensional lattice. The vortex container is the set of all lattice sites which are enclosed by the two closed loops depicted as dashed lines, i.e. the set of all crosses $+$, circles $\bullet$ and squares $\blacksquare$. The interior Λ of the container T is the set of all crosses $+$, whereas the boundary ∂T consists of all circles $\bullet$ (= the outer boundary ∂T^{out}) and squares $\blacksquare$ (= the inner boundary ∂T^{in}).

container T_i are omitted. The inner integral equals the partition function $Z(T_i, \bar{U}_x)$ defined on the container T_i with boundary conditions $\bar{U}_x,\ x \in \partial T_i$,

$$Z(T_i, \bar{U}_x) = \int \prod_{x \in T_i} D\mu(U'_x) \exp\left[\beta {\sum_{l \in T_i}}' \operatorname{ReTr}(U'_x U'^{\dagger}_{x+n})\right] \prod_{x \in \partial T_i} \delta(U'_x \bar{U}_x^{-1}). \tag{5}$$

An important property of $Z(T_i, \bar{U}_x)$ is the invariance under $SU(N)$ transformations of the boundary conditions. In particular, $Z(T_i, \bar{U}_x)$ remains unchanged under the transformation

$$\bar{U}_x \;\rightarrow\; \omega_i^{-1} \bar{U}_x, \qquad x \in \partial T_i, \tag{6}$$

with ω_i an element of the center $Z(N)$ of $SU(N)$. Let us perform now the following variable

substitution

$$
\begin{aligned}
\bar{U}_x &= U_x \prod_{k=i}^{N_c} \omega_k && \text{if} \quad \Lambda_{i-1} < x < \Lambda_i \\
&= U_x \prod_{k=1}^{N_c} \omega_k && \qquad\qquad x < \Lambda_1 \\
&= U_x && \qquad \Lambda_{N_c} < x.
\end{aligned} \tag{7}
$$

The notation $\Lambda_{i-1} < x < \Lambda_i$ means that the lattice site x belongs to the region which is enclosed by the interior Λ_{i-1} of the container T_{i-1} and the interior Λ_i of the container T_i. N_c is the total number of containers. In the special case of $\bar{U}_0$ and $\bar{U}_R$, the substitution (7) results in $\bar{U}_0 = U_0 \prod_i \omega_i$ and $\bar{U}_R = U_R$. For the correlation function (4) we can write

$$
\Gamma_R(\beta) = \frac{1}{Z} \int \prod_{x \in \Lambda_c} D\mu(U_x) \, \mathrm{ReTr}(U_0 U_R^\dagger) \, \exp[\beta \sum_{l \in \Lambda_c} \mathrm{ReTr}(U_x U_{x+n}^\dagger)] \, \prod_i \omega_i Z(T_i, U_x^\omega) \tag{8}
$$

with boundary conditions U_x^ω on the container T_i

$$
\begin{aligned}
U_x^\omega &= U_x \prod_{k=i+1}^{N_c} \omega_k && \text{if} \quad x \in \partial T_i^{\mathrm{in}} \\
&= U_x \, \omega_i^{-1} \prod_{k=i+1}^{N_c} \omega_k && \qquad x \in \partial T_i^{\mathrm{out}}.
\end{aligned} \tag{9}
$$

$\partial T_i^{\mathrm{in}}$ and $\partial T_i^{\mathrm{out}}$ are the inner (the set of squares $\blacksquare$ in fig. 1) and outer boundary (the set of circles $\bullet$ in fig. 1) of the container T_i respectively. We further simplify the boundary conditions (9) by applying the transformation $U_x \to U_x \, \omega_i \prod_{k=i+1} \omega_k^{-1}$ under which $Z(T_i, U_x^\omega)$ remains unchanged. Thus we finally can write

$$
\begin{aligned}
U_x^\omega &= U_x \, \omega_i && \text{if} \quad x \in \partial T_i^{\mathrm{in}} \\
&= U_x && \qquad x \in \partial T_i^{\mathrm{out}}
\end{aligned} \tag{10}
$$

for the boundary conditions U_x^ω of the container T_i. Since $\omega_i \in Z(N)$ is arbitrary, we may sum (integrate) over ω_i in (8) using the normalized Haar measure on $Z(N)$. Using the trivial relations

$$
\begin{aligned}
\left| \mathrm{ReTr}(U_0 U_R^\dagger) \right| &\leq \mathrm{Tr}(\mathbf{1}) \\
\left| \frac{\sum_{\omega_i} \omega_i Z(T_i, U_x^\omega)}{\sum_{\omega_i} Z(T_i, U_x^\omega)} \right| &\leq \max_{U_x} \left| \frac{\sum_{\omega_i} \omega_i Z(T_i, U_x^\omega)}{\sum_{\omega_i} Z(T_i, U_x^\omega)} \right|
\end{aligned} \tag{11}
$$

and with the help of the identity

$$
Z = \int \prod_{x \in \Lambda_c} D\mu(U_x) \exp[\beta \sum_{l \in \Lambda_c} \mathrm{ReTr}(U_x U_{x+n}^\dagger)] \, \prod_i \sum_{\omega_i} Z(T_i, U_x^\omega) \tag{12}
$$

we derive the following bound for the correlation function (8)

$$|\Gamma_R(\beta)| \;\leq\; \mathrm{Tr}(\mathbf{1}) \prod_i \max_{U_x} \left| \frac{\sum_{\omega_i} \omega_i Z(T_i, U_x^{\omega})}{\sum_{\omega_i} Z(T_i, U_x^{\omega})} \right| . \tag{13}$$

The maximum on the right hand side of (13) is to be understood as the maximum under the boundary conditions (10). In order to make this result more transparent we will specify it for the case of $SU(2)$. Then ω_i can take values ± 1 and the bound (13) can be written

$$|\Gamma_R(\beta)| \;\leq\; \mathrm{Tr}(\mathbf{1}) \prod_i \max_{U_x} |V_i(T_i, U_x^{\omega})| \,, \tag{14}$$

where we introduced

$$V_i(T_i, U_x^{\omega}) \;=\; \frac{1-q_i}{1+q_i}. \tag{15}$$

q_i is called the vortex free energy and is given by the ratio of partition functions defined on the vortex container T_i with boundary conditions U_x^{ω} (10)

$$q_i = \frac{Z(T_i, U_x^{\omega=-1})}{Z(T_i, U_x^{\omega=1})}. \tag{16}$$

The physical meaning of q_i (and more generally of V_i) is rather obvious: q_i measures the change of free energy $\exp(-\Delta F)$ introduced by a singular transformation $U_x \to U_x \omega_i$ on one of the boundaries of the container T_i. Let L_i be the diameter of the container T_i. Then we say that the transformation $U_x \to U_x \omega_i$ performed on one of the boundaries of the container introduces a vortex of thickness L_i into the system and q_i measures the energy needed to create such a vortex.

We are now ready to explain what we call vortex condensation mechanism for generating a non-zero MG in $2D$ $SU(N)$ spin models. Let V_i^{max} be the maximum of $|V_i|$ under the boundary conditions U_x^{ω} defined in (10); i.e. $V_i^{max} = \max_{U_x} |V_i|$. Suppose that there exists such a mass m_v that for each container T_i and for sufficiently large diameter L_i, the maximum V_i^{max} behaves according to

$$V_i^{max} \;\sim\; \exp(-m_v(\beta) L_i). \tag{17}$$

Then, with $R = \sum_i L_i$, the bound (14) for the correlation function reads

$$|\Gamma_R(\beta)| \;\leq\; \mathrm{const} \cdot \exp(-m_v(\beta) R), \tag{18}$$

which is the expected exponential decay. We term m_v vortex MG to distinguish it from the genuine MG m_c extracted from the correlation function. If $m_v = m_c$, condensation of vortices can be made responsible for generating a non-zero MG in $2D$ $SU(N)$ spin models.

The crucial quantity in a vortex condensation mechanism is the vortex free energy q_i which is originally defined on the container T_i but which may be calculated on any lattice having the same topology as the container; i.e. the topology of a ring. Thus, we have to consider a lattice with periodic boundary conditions in one direction and fixed boundary conditions in the other

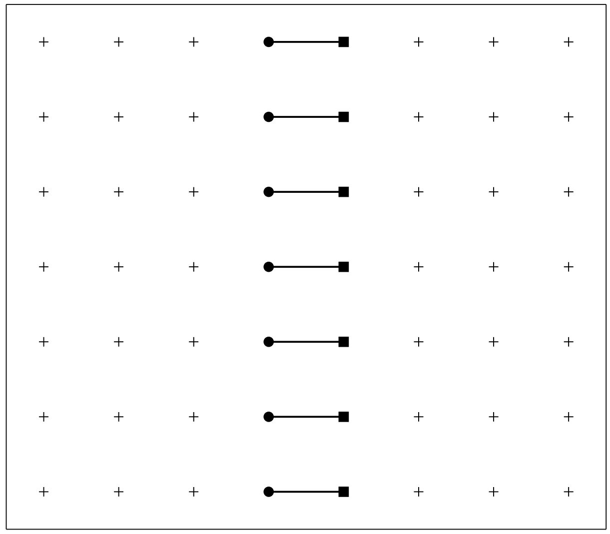

Figure 2: A lattice (periodically closed) which has the same topology as the container shown in fig. 1. Spins defined on circles • and squares ▪ are fixed to given values. In the case of the vortex condensation mechanism introduced in ref. [2], the coupling β is changed to $-\beta$ on the depicted links.

direction. A corresponding lattice is shown in fig. 2. The vortex free energy q on this lattice is given by

$$q = \frac{Z(U_x\,\omega)}{Z(U_x)}, \tag{19}$$

where in the denominator values U_x are assigned to spins depicted as circles • and squares ▪ in fig. 2, while in the numerator spins living either on circles or on squares are fixed to values $U_x\omega$, with $\omega \in Z(2)$. The partition function Z is defined in (5).

Finally, we think that still some comments are needed at this place:

- Configurations which rotate slowly from one center element to another one on some characteristic length scale are termed "thick vortices". According to the paper by Dobrushin and Shlosman [15] they produce disordering effects which are sufficient to enforce the long distance correlation function to fall off to zero at any coupling; i.e. they guarantee the absence of magnetization. However, a priori there is no reason to believe that thick vortices provide an *exponential* fall off. It might be that for $\beta \to \infty$ these configurations can only account for a power law decay of the correlation function. Two conclusions can be drawn from such a scenario. 1) Thick vortices are not responsible for a non-zero MG at arbitrarily large values of β or 2) they are at small β but the system undergoes a phase transition to a massless phase at a finite value of β.

- The above vortex condensation theory does not give an exact definition of a vortex. The only important issue is a change of vorticity over some characteristic length scale. To specify a vortex completely, one has to define the rate of this change and, possibly, the dependence of the characteristic length scale on the bare coupling β. It is reasonable

to assume that on this length scale, the $SU(N)$ spin model looks like an effective $Z(N)$ model for special $Z(N)$ excitations which one should be able to extract from the original configurations.

2.1 $1D$ model

Before finishing this section we will demonstrate on a simple example how the mechanism described above works in practice. The simplest example we can imagine is the $SU(2)$ spin model in one dimension where it is well known that the correlation function shows an exponential fall off at any value of the coupling constant β and the corresponding MG is given by

$$m_c(\beta) = \ln\left(\frac{I_1(\beta)}{I_2(\beta)}\right), \tag{20}$$

with I_n modified Bessel functions. In one dimension a vortex container is simply a chain of spins with Dirichlet-like boundary conditions. More precisely: let L be the length of the $1D$ chain and let us fix the spins on the boundary to some arbitrary value

$$U_{x=0} = W_1, \quad U_{x=L} = W_2. \tag{21}$$

Define $W = W_1 W_2^\dagger$. On a finite lattice, the partition function Z can be calculated exactly

$$Z(W) = \beta^{-L} \sum_{n=0,1/2,\ldots}^{\infty} (2n+1)\chi_n(W)\left[I_{2n+1}(\beta)\right]^L, \tag{22}$$

where $\chi_n(W)$ is the character of the n^{th} representation of $SU(2)$. Perform now a nontrivial $Z(2)$ transformation on one of the boundary spins of our chain, e.g. $W_2 \to \omega W_2 = -W_2$. The character $\chi_n(W)$ transforms according to

$$\chi_n(W) \to \chi_n(-W) = (-1)^{2n}\chi_n(W). \tag{23}$$

For the vortex free energy q (16) we obtain

$$q = \frac{Z(-W)}{Z(W)} \approx 1 - 4\chi_{1/2}(W)\left(\frac{I_2(\beta)}{I_1(\beta)}\right)^L + \ldots, \tag{24}$$

where the terms which vanish faster in the thermodynamic limit have been neglected. The maximum of V (14) occurs at $W = \mathbf{1}$ and we finally find

$$\max_W V \;\sim\; \exp[-L\ln\left(\frac{I_1(\beta)}{I_2(\beta)}\right)]. \tag{25}$$

Thus, in the $1D$ model the change of V introduced by the creation of a thick vortex shows an exponential decrease with a MG which equals the MG extracted from the correlation function (20). Of course, in one dimension this result is in a sense trivial but it shows nevertheless transparently how the idea described above works.

3 Effective model for $Z(2)$ excitations

In this section we address the question which are the spin field configurations playing a crucial role in a vortex condensation theory. In the case of gauge theories, it was suggested long ago that certain $Z(N)$ excitations of the gauge field are responsible for an area law behaviour of the Wilson loop [14]. Here, we present an effective model for $Z(2)$ excitations in the $2D$ $SU(2)$ spin model and establish a link to the vortex condensation mechanism. The main assumption of this model is supported by MC data which will be presented in next Section.

As a first step we fix proper boundary conditions of the two-dimensional lattice. It is convenient to take a periodic lattice in y-direction and to fix the $SU(2)$ spins to elements of the $Z(2)$ subgroup if $x = 0$ and $x = L$. L is the linear extension of the lattice. It should be stressed that fixing boundary conditions is not necessary for systems with global symmetry since the Mermin-Wagner theorem guarantees independence of the results on BC in the infinite volume limit. In general, a dependence on BC should vanish faster than any long-distance correlation function. In our case fixing boundary conditions is only a question of convenience and proper definitions.

We start by rewriting the partition function of the $SU(N = 2)$ principal chiral model (1) using the representation

$$U_x \; = \; z_x \bar{U}_x \tag{26}$$

for the spin field $U_x \in SU(2)$, where z_x is a $Z(2)$ element and $\bar{U}_x \in SU(2)/Z(2)$. In the new variables the BC for z_x are free ones while for $\bar{U}_x$ we have to impose Dirichlet BC in x-direction. For the invariant measure on the $SU(2)$ group we write

$$D\mu(U_x) = \frac{1}{2} \sum_{\{z_x\}=\pm 1} D\mu(\bar{U}_x), \tag{27}$$

where $D\mu(\bar{U}_x)$ is an invariant measure on the $SO(3)$ group. Let us recall that the invariant measure on the $SU(N)/Z(N)$ group coincides with the $SU(N)$ measure up to the restriction

$$-\frac{2\pi}{N} \leq \arg(\mathrm{Tr} U_x) \leq \frac{2\pi}{N}. \tag{28}$$

In other words, in the invariant $\bar{U}_x$-integration the trace of the fundamental characters is restricted to positive values. Performing now the summation over the $Z(2)$ elements z_x one can rewrite the partition function (1) as (up to an irrelevant constant)

$$Z(\beta) = \sum_{\{s_l\}=\pm 1} \sum_{\mathcal{L}} \int \prod_x D\mu(\bar{U}_x) \exp[\, \beta \sum_l s_l \mathrm{Tr}(\bar{U}_x \bar{U}^{\dagger}_{x+n})] \prod_{l \in \mathcal{L}} s_l. \tag{29}$$

$\mathcal{L}$ is a set of closed loops and we have introduced a new $Z(2)$ link variable s_l. This representation for $Z(\beta)$ (and a similar one in the case of $SU(2)$ gauge theory) was derived and investigated in [16]. In terms of the link variable s_l the singular transformations discussed in section 2 may be defined as a change of the sign of s_l on the boundaries of a vortex container T (only in the action). The sum over $\mathcal{L}$ in (29) is a sum over all closed loops (including all their possible products) taken with an appropriate weight. It is defined exactly in the same way as in the two-dimensional Ising model where much is known about the properties of such a loop expansion.

However, unlike the Ising model, the coefficients defined on the links of closed loops are not constant. Moreover, if we consider the model (29) as an Ising-like model with a fluctuating coupling constant, we observe that the coupling is not positive definite (despite $\mathrm{Tr}\bar{U}_x > 0$ this is not the case for the character of the product of two group elements). At least at first glance, this could imply that fluctuations of the link variable s_l may persist down to weak coupling and cause the disorder which is needed for an exponential behaviour of the correlation function. In fact, this is precisely what we are going to work out.

As a first step we perform the sum over the link variables s_l. It results in

$$Z(\beta) = \sum_{\mathcal{L}} \int \prod_x D\mu(\bar{U}_x) \prod_l \cosh[\,\beta\,\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})] \prod_{l\in\mathcal{L}} \tanh[\,\beta\,\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})]. \tag{30}$$

For sufficiently large values of β we obtain

$$Z(\beta) = \sum_{\mathcal{L}} (\tanh 2\beta)^{|\mathcal{L}|} \int \prod_x D\mu(\bar{U}_x) \prod_l \exp[\,\beta\,|\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})|\,] \prod_{l\in\mathcal{L}} \sigma_l \;+ \mathcal{O}(e^{-4\beta}), \tag{31}$$

where

$$\sigma_l = \mathrm{sign}[\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})]. \tag{32}$$

¿From the last equations we see that at large values of β the original partition function, and so the free energy, can be written as a product of two partition functions - the partition function of a $SU(2)/Z(2)$ model and the partition function of an Ising-like model. Precisely

$$Z(\beta) = Z^{SU(2)/Z(2)} Z^I, \tag{33}$$

with

$$Z^{SU(2)/Z(2)} = \int \prod_x D\mu(\bar{U}_x) \prod_l \exp[\,\beta\,|\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})|\,] \tag{34}$$

and the Ising-like partition function

$$Z^I = \sum_{\mathcal{L}} (\tanh 2\beta)^{|\mathcal{L}|} F(\mathcal{L}). \tag{35}$$

$F(\mathcal{L})$ is defined as an expectation value

$$F(\mathcal{L}) = \langle \prod_{l\in\mathcal{L}} \sigma_l \rangle_{SU(2)/Z(2)} \tag{36}$$

which has to be evaluated in the ensemble (34). In the case of the fundamental correlation function (2) for $N = 2$ we follow the same strategy and obtain for large β-values

$$\Gamma_R(\beta) = (Z^{SU(2)/Z(2)} Z^I)^{-1} \sum_{\mathcal{P}} (\tanh 2\beta)^{|\mathcal{P}|} \sum_{\mathcal{L}/l\in\mathcal{P}} (\tanh 2\beta)^{|\mathcal{L}|} \int \prod_x D\mu(\bar{U}_x)\, \mathrm{Tr}(\bar{U}_0 \bar{U}^\dagger_R) \;\times$$
$$\exp[\,\beta\,|\mathrm{Tr}(\bar{U}_x \bar{U}^\dagger_{x+n})|\,] \prod_{l\in\mathcal{P}} \sigma_l \prod_{l\in\mathcal{L}} \sigma_l \;+\; \mathcal{O}(e^{-4\beta}). \tag{37}$$

Here, the sum over $\mathcal{P}$ is a sum over all paths (including all their possible products) connecting the lattice sites 0 and R. In addition, one has to sum up over all possible closed loops on the lattice which must not have a link in common with a given path $\mathcal{P}$. Again, this is in full accordance with the expansion of the correlation function in the Ising model. To see this explicitly, let us write down the corresponding formulae for the Ising model. For the partition function one gets an expansion in terms of loops (up to a constant)

$$Z^{Ising} = \sum_{\mathcal{L}} (\tanh\beta)^{|\mathcal{L}|}, \tag{38}$$

while the correlation function reads

$$\Gamma_R^{Ising}(\beta) = (Z^{Ising})^{-1} \sum_{P} (\tanh\beta)^{|\mathcal{P}|} \sum_{\mathcal{L}/l\in\mathcal{P}} (\tanh\beta)^{|\mathcal{L}|}. \tag{39}$$

These formulae have to be compared with the asymptotic expansions (31) and (37) in the case of the $SU(2)$ model. In the Ising model it is known that for β-values below β_c the correlation function is well approximated by the expression

$$\Gamma_R^{Ising}(\beta < \beta_c) \approx \sum_{\mathcal{P}_c} (\tanh\beta)^{|\mathcal{P}_c|}, \tag{40}$$

where we have to take into account only the so-called connected graphs $\mathcal{P}_c$. In the following we shall show that one can arrive at a similar expression for the correlation function in the $SU(2)$ model by making suitable assumptions. Firstly, we introduce a $Z(2)$ correlation function in the $SU(2)$ ensemble according to

$$\Gamma_R^{Z(2)}(\beta) = \langle z_0 z_R \rangle \tag{41}$$

with $z_x = \mathrm{sign}(\mathrm{Tr}U_x)$ and claim that it reproduces the correct long distance behaviour of the full $SU(2)$ model, i.e. the MG extracted from (41) coincides with the full $SU(2)$ MG. This assumption will be confirmed in the next section by numerical results. It leads to the following expression for the asymptotic expansion of the $SU(2)$ correlation function (37)

$$\Gamma_R(\beta) = \frac{1}{Z^I} \sum_{\mathcal{P}} (\tanh 2\beta)^{|\mathcal{P}|} \sum_{\mathcal{L}/l\in\mathcal{P}} (\tanh 2\beta)^{|\mathcal{L}|} \langle \prod_{l\in\mathcal{P}} \sigma_l \prod_{l\in\mathcal{L}} \sigma_l \rangle_{SU(2)/Z(2)}, \tag{42}$$

where the expectation value is defined in the ensemble (34). Our next assumption is that for β-values smaller than β_c (42) can be written as a sum over connected graphs like in the Ising model

$$\Gamma_R(\beta < \beta_c) \approx \sum_{\mathcal{P}_c} (\tanh 2\beta)^{|\mathcal{P}_c|} \langle \prod_{l\in\mathcal{P}_c} \sigma_l \rangle_{SU(2)/Z(2)}. \tag{43}$$

This formula has to be compared with the corresponding formula (40) in the case of the Ising model. In the $SU(2)$ case the conventional scenario means $\beta_c = \infty$. Thus we expect that approximation (43) works rather well in the whole region of the bare coupling. Of course, its goodness is determined by the behaviour of the expectation value on the right hand side of (43). Suppose for a while that $\langle \prod_{l\in\mathcal{P}_c} \sigma_l \rangle_{SU(2)/Z(2)}$ is small enough to cancel the fast growing number

of connected paths with length $|\mathcal{P}_c|$ and to guarantee a fast convergence on the right-hand side of (43). Then, it is easy to give a prediction for the MG. To leading order one finds for $\beta < \beta_c$

$$m_{eff}(\beta < \beta_c) \approx -\frac{1}{R} \ln \left[(\tanh 2\beta)^R \, \langle \prod_{l \in \mathcal{P}_{min}} \sigma_l \, \rangle_{SU(2)/Z(2)} \right] , \tag{44}$$

where $\mathcal{P}_{min}$ is the shortest path between the sites 0 and R. Next to leading order corrections can be obtained from (43).

Let us discuss now some of the issues involved with the above Ising-like effective model as well as some possible physical scenarios.

1. A link of the above effective model to the vortex condensation theory presented in the previous section can be established as follows: In analogy to the $Z(2)$ correlation function let us define the $Z(2)$ vortex free energy. To do this, we rewrite the ratio of partition functions q defined in (19) as an expectation value of an appropriate operator which can be expanded in a sum over $SU(2)$ representations. Making use of the decomposition (26) - (28) and proceeding along the same line as in the case of the correlation function we are able to express the vortex free energy in terms of expectation values defined in (36) and (43). Thus, if it turns out that condensation of vortices is responsible for a non-zero MG and this MG can be extracted from (37) and (43) respectively, then the $Z(2)$ degrees of freedom reproduce the exponential decay of the vortex free energy as well. However, this is only possible in a $SU(2)/Z(2)$ background, since the $Z(2)$ degrees of freedom alone cannot account for an exponential fall off.

2. If $Z(2)$ degrees of freedom indeed play a crucial role in generating a non-zero MG, this may have some strong impact to $SU(2)/Z(2)$ models. Consider for example the following variant of an $SO(3)$ model which is an analogue of the lattice gauge model introduced in [18]

$$\begin{aligned} Z^{SO(3)} &= \sum_{\{s_l\}=\pm 1} \int \prod_x D\mu(U_x) \exp[\, \beta \sum_l s_l \, \mathrm{Tr}(U_x U_{x+n}^\dagger)\,] = \\ &= \int \prod_x D\mu(U_x) \prod_l \cosh[\, \beta \, \mathrm{Tr}(U_x U_{x+n}^\dagger)\,]. \end{aligned} \tag{45}$$

At large β we obtain

$$Z^{SO(3)} = \int \prod_x D\mu(U_x) \prod_l \exp[\, \beta \, |\mathrm{Tr}(U_x U_{x+n}^\dagger)|]. \tag{46}$$

Speculations about the coincidence of $SO(3)$ and $SU(2)$ models are based on the widely accepted belief that the continuum limit has to be taken at $\beta \to \infty$, where the naive continuum limits of both models coincide. But since the $SO(3)$ model (46) lacks of $Z(2)$ degrees of freedom, a possible non-zero MG cannot be explained with the above described effective model. Thus, either $SU(2)$ and $SO(3)$ models have different continuum limits, or the Ising-like part (35) of the $SU(2)$ partition function (33) as well as the correlation function (43) must become trivial in this limit. The latter scenario means that the MG

extracted from the $Z(2)$ degrees of freedom vanishes above some pseudo-critical value of β while the full $SU(2)$ MG does not. This seems to be very unlikely, since at least for finite values of β the $Z(2)$ correlation function carries the whole MG. This will be shown by numerical results in the next section.

3. Suppose for a moment that the $Z(2)$ correlation function indeed reproduces the full MG. Comparing formulae (35) and (38) one can determine an effective Ising coupling in the $SU(2)$ model. Two scenarios are possible. If the effective coupling is less than the critical coupling of the Ising model for arbitrarily large values of the bare coupling β, the $SU(2)$ model is always in a phase with a non-zero MG and the continuum limit may be taken at $\beta \to \infty$ according to the conventional scenario (at the same time the effective coupling has to approach the critical coupling of the Ising model, otherwise the very existence of a nontrivial continuum limit becomes problematic). It may happen, however, that at some large value of β the effective Ising coupling becomes larger than the critical coupling of the Ising model. Then above this β-value the system is in a massless phase and the conventional scenario is broken since one has to realize the continuum limit at this finite β-value corresponding to the critical effective Ising coupling. It is interesting to mention that in this case if it is possible to construct the massless continuum limit by driving the bare coupling to its critical value from above, one may expect that this limit coincides with the continuum limit of the $SO(3)$ model.

4 Monte-Carlo study of $Z(2)$ and $SU(2)$ mass gap

In this section we will show by results of MC simulations that the long distance behaviour of the full $SU(2)$ correlation function (2) coincides with the long distance behaviour of the $Z(2)$ correlation function (41). This is done by calculating the full $SU(2)$ correlation length $\xi_{SU(2)}$ (which is the inverse of the MG m) and by comparing it with the correlation length $\xi_{Z(2)}$ extracted from $Z(2)$ degrees of freedom. As will be seen, both quantities - $\xi_{SU(2)}$ and $\xi_{Z(2)}$ - agree within errorbars. This result confirms the assumption made in the previous section that the $Z(2)$ degrees of freedom carry the full $SU(2)$ MG. Furthermore, we consider the PL model and demonstrate that $Z(2)$ excitations reproduce the full $SU(2)$ MG in this model as well.

4.1 Standard model

To calculate the full $SU(2)$ correlation length $\xi_{SU(2)}$ we follow the method presented in [19]†. We parameterize our field variables $U_x \in SU(2)$ according to

$$U_x \;=\; u_0(x) + i\vec{u}(x)\vec{\sigma}, \qquad\qquad u_x \;=\; (u_0(x), \vec{u}(x)), \tag{47}$$

†To be precise, in ref. [19] the correlation length ξ is defined as the second-moment correlation length and not as the inverse of the mass gap m which can be determined by fitting the falloff of the zero-momentum correlation function to the cosh-behaviour appropriate for time-periodicity. However, both quantities are expected to show the same scaling behaviour. Surprisingly, it is found empirically that the two definitions of ξ do not only scale in the same way but agree within less than 1% [19, 20]. In this article we employ the definition of ξ presented in [19] since it is less CPU-time consuming and we feel free to call it the inverse of the MG.

with $\vec{\sigma}$ the Pauli matrices. We then determine the susceptibility $\chi_{SU(2)}$ according to

$$\chi_{SU(2)} = \frac{1}{V} \langle \, (\sum_x u_x)^2 \, \rangle \tag{48}$$

and the analogous quantity at the smallest non-zero momentum

$$F_{SU(2)} = \frac{1}{V} \langle \frac{1}{2} \left[|\sum_x e^{2\pi i x_1/L} u_x|^2 + |\sum_x e^{2\pi i x_2/L} u_x|^2 \right] \rangle. \tag{49}$$

In terms of these two quantities the second-moment correlation length $\xi_{SU(2)}$ is given by [19]

$$\xi_{SU(2)} = \left(\frac{\chi_{SU(2)}/F_{SU(2)} - 1}{4 \sin^2 \pi/L} \right)^{1/2} . \tag{50}$$

L is the linear size of the lattice and $V = L^2$ the number of sites in the lattice. In addition, we calculate the internal energy

$$E_{SU(2)} = \Gamma_1(\beta) = \langle \mathrm{Tr}(U_0 U_1^\dagger) \rangle \tag{51}$$

corresponding to the correlation function of spins separated by one lattice spacing. In the case of the $Z(2)$ correlation length $\xi_{Z(2)}$ we consider the same ensemble of $SU(2)$ degrees of freedom U_x but instead of calculating the quantities χ (48) and F (49) with the parameters u_x (47) we use the $Z(2)$ degrees of freedom

$$z_x = \mathrm{sign}(\mathrm{Tr} U_x) = \mathrm{sign}(u_0(x)). \tag{52}$$

$\xi_{Z(2)}$ is then determined by inserting $\chi_{Z(2)}$ and $F_{Z(2)}$ into (50) and $E_{Z(2)}$ is given by $\langle z_0 z_1 \rangle$.

In order to simulate the $2D$ $SU(2)$ principal chiral model defined in (1) with $N = 2$ we use Wolff's cluster algorithm [21]. The model is considered at different values of the inverse coupling constant β. We choose periodic boundary conditions. In all cases we start with a random configuration and apply at least 10^4 warm up sweeps. We generate $5 \cdot 10^5$ configurations and measure the quantities of interest in every configuration since the cluster algorithm is known to show small autocorrelation times. To estimate errorbars we use the jackknife sub-ensemble analysis. In table 1 we show the results for $\xi_{SU(2)}$ and $\xi_{Z(2)}$ for several β-values and lattice sizes $L = 128$ and $L = 256$. The data in the column denoted with ξ_E is taken from [19] and serves as reference results for $\xi_{SU(2)}$. As is clearly seen from table 1 the $SU(2)$ correlation length $\xi_{SU(2)}$ and the $Z(2)$ correlation length $\xi_{Z(2)}$ show perfect agreement within errorbars. This indicates that the $Z(2)$ degrees of freedom alone carry the full $SU(2)$ MG.

For comparison we also computed the internal energy E (51) which corresponds to the correlation function of spins separated by one lattice spacing. At such a small distance and for large enough values of β the spin system is well ordered. According to a vortex condensation theory this means that thin vortices are suppressed and the spin configurations are not $Z(2)$-like at short distances. Thus, we do *not* expect agreement between the full $SU(2)$ internal energy $E_{SU(2)}$ and the internal energy $E_{Z(2)}$ extracted from $Z(2)$ degrees of freedom. Table 2 shows numerical results which suggest that this is indeed the case: There is a large discrepancy between $E_{SU(2)}$ and $E_{Z(2)}$ which indicates that the $Z(2)$ degrees of freedom cannot account for the short distance behaviour of the $SU(2)$ model.

L	2β	$\xi_{SU(2)}$	$\xi_{Z(2)}$	ξ_E
128	2.2	14.01 (0.16)	14.36 (0.24)	14.02 (0.03)
	2.3	18.95 (0.13)	18.66 (0.19)	18.91 (0.05)
	2.4	25.39 (0.13)	25.10 (0.18)	25.15 (0.07)
	2.5	33.14 (0.12)	32.95 (0.17)	33.19 (0.07)
	2.6	41.77 (0.12)	41.32 (0.19)	41.38 (0.09)
	2.7	49.44 (0.13)	49.61 (0.19)	49.60 (0.11)
256	2.4	26.02 (0.37)	25.23 (0.60)	25.50 (0.20)
	2.5	34.51 (0.28)	34.63 (0.43)	34.97 (0.16)
	2.6	46.44 (0.25)	46.86 (0.38)	46.66 (0.17)
	2.7	62.06 (0.25)	62.28 (0.37)	61.90 (0.23)
	2.8	78.20 (0.26)	78.55 (0.37)	78.48 (0.27)

Table 1: Estimates for the correlation length $\xi_{SU(2)}$ extracted from $SU(2)$ degrees of freedom and for $\xi_{Z(2)}$ determined with the $Z(2)$ degrees of freedom. The data in the right column denoted by ξ_E show results for $\xi_{SU(2)}$ taken from [19].

L	2β	$E_{SU(2)}$	$E_{Z(2)}$	E_E
128	2.2	1.24528 (0.00027)	0.92359 (0.00062)	1.24509 (0.00001)
	2.3	1.28420 (0.00018)	0.95440 (0.00044)	1.28394 (0.00001)
	2.4	1.31898 (0.00016)	0.98440 (0.00043)	1.31886 (0.00001)
	2.5	1.35023 (0.00011)	1.01017 (0.00042)	1.35029 (0.00001)
	2.6	1.37882 (0.00009)	1.03454 (0.00038)	1.37874 (0.00001)
	2.7	1.40463 (0.00007)	1.05722 (0.00042)	1.40460 (0.00001)
256	2.4	1.31541 (0.00358)	0.98089 (0.00237)	
	2.5	1.34857 (0.00171)	1.00920 (0.00128)	1.35021 (0.00001)
	2.6	1.37834 (0.00015)	1.03407 (0.00034)	1.37860 (0.00001)
	2.7	1.40443 (0.00006)	1.05648 (0.00032)	1.40443 (0.00001)
	2.8	1.42805 (0.00005)	1.07736 (0.00031)	1.42806 (0.00001)

Table 2: Estimates for the internal energy $E_{SU(2)}$ extracted from $SU(2)$ degrees of freedom and for $E_{Z(2)}$ determined with the $Z(2)$ degrees of freedom. The data in the right column denoted by E_E show results for $E_{SU(2)}$ taken from [19].

Finally let us compare our results obtained in the $2D$ $SU(2)$ principal chiral model with similar results obtained in $4D$ gauge theories. In gauge theories, it was shown by numerical simulations that the asymptotic ST is nearly reproduced by $U(1)$ and $Z(2)$ degrees of freedom respectively. While $U(1)$ dominance [22] is in favour of the dual superconductor picture of confinement, $Z(2)$ dominance [11] indicates that a vortex condensation mechanism is responsible for confinement of static quarks. However, in both cases the gauge needs to be partially fixed to extract the relevant degrees of freedom, and this procedure is apparently ambiguous. It was reported for example that the Abelian $U(1)$ dominance is convincingly observed only in one particular gauge, the so-called maximal Abelian gauge. We want to emphasize that in the $2D$ $SU(2)$ principal chiral model considered in this paper, there is no gauge freedom and thus, there is no need to fix the gauge. The determination of the $Z(2)$ degrees of freedom (52) is unambiguous and the same is true for the results presented in tables 1 and 2.

4.2 Positive link model

The numerical results of the previous section suggest that the $Z(2)$ degrees of freedom alone reproduce the long distance behaviour of the $SU(2)$ correlation function while this is not the case at short distances. This observation is in favour of a vortex condensation theory according to which the $SU(2)$ spin field configurations behave $Z(2)$-like only at large distances. In this section we investigate the dependence of this mechanism on the short distance structure of the $SU(2)$ model. We restrict the trace of the link variables to positive values and call this model the positive link model (PLM). According to the conventional scenario the continuum limit of the $SU(2)$ model has to be taken at $\beta \to \infty$. Thus, the PLM and the standard $SU(2)$ model have the same continuum limit. Moreover, a possible mechanism which is responsible for the existence of a non-zero MG should be the same in both models.

The positive link model is defined as an analogue of the positive plaquette model in gauge theories [8]. It restricts the trace of link variables to positive values and thus suppresses thin vortices of the order of one lattice spacing. Such thin vortices can be seen as lattice artifacts. They should not influence the above discussed vortex condensation mechanism which is based on condensation of thick vortices having a linear extension of many lattice spacings. In this sense, the PLM is closer to the continuum than the standard $SU(2)$ model. The action of the PLM can be written as

$$S_{PLM} \; = \; \beta \sum_{x,n} \mathrm{Tr}(U_x U_{x,n}^\dagger) \; - \; \lambda \sum_{x,n} [1 - \mathrm{sign}(\mathrm{Tr}(U_x U_{x,n}^\dagger))], \tag{53}$$

where for a complete suppression of negative links we have to choose $\lambda = \infty$. To simulate the partition function $Z_{PLM} = \int D\mu(U_x) \exp(S_{PLM})$ of the PLM we use a heatbath algorithm. In the update the change of a $SU(2)$ spin variable is rejected, if the trace of one or more of the resulting four links is negative. Fortunately, the rejection rate decreases with increasing values of β. Simulations were run on a lattice with size $L = 128$ and at four different β-values. In all cases we started with the trivial configuration and applied $2 \cdot 10^4$ warm up sweeps. We then generated $5 \cdot 10^6$ configurations and measured the quantities of interest in every fifth configuration. It should be emphasized that in this paper we are not interested in the question of scaling in the PLM. Our aim is to find out whether the $Z(2)$ degrees of freedom carry the full $SU(2)$ MG in the PLM as well. We thus calculated the correlation lengths $\xi_{SU(2)}$ (50) and $\xi_{Z(2)}$ as introduced in the previous section. The numerical results are shown in table 3. It is clearly seen that at

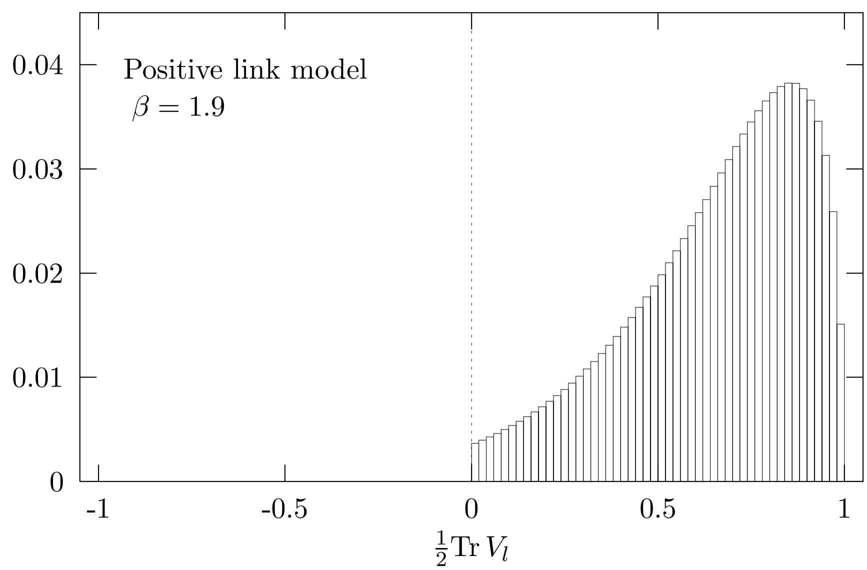

Figure 3: Distribution of link traces in the positive link model.

L	2β	$\xi_{SU(2)}$	$\xi_{Z(2)}$	$E_{SU(2)}$	$E_{Z(2)}$
128	1.7	20.24 (0.19)	19.17 (0.40)	1.27350 (0.00001)	0.93670 (0.00032)
	1.8	23.53 (0.21)	23.92 (0.46)	1.29278 (0.00001)	0.95381 (0.00043)
	1.9	26.93 (0.27)	27.24 (0.55)	1.31180 (0.00001)	0.97044 (0.00058)
	2.0	31.63 (0.27)	31.63 (0.94)	1.33056 (0.00001)	0.98686 (0.00079)

Table 3: Estimates for the correlation length ξ and the internal energy E extracted from both $SU(2)$ and $Z(2)$ degrees of freedom in the positive link model.

least for large enough values of β the two quantities agree within errorbars. This shows that in the PLM the $Z(2)$ degrees of freedom reproduce the long distance behaviour of the $SU(2)$ correlation function as well. We interpret this result as an indication that also at weak coupling condensation of thick vortices is the mechanism which leads to the existence of a non-zero MG. For comparison we computed the internal energy $E_{SU(2)}$ (51). For reasons given in the previous section we do not expect that the $SU(2)$ internal energy $E_{SU(2)}$ and the internal energy $E_{Z(2)}$ extracted from $Z(2)$ degrees of freedom agree. Table 3 shows that this is indeed the case.

5 Summary

In this article we discussed a vortex condensation mechanism to explain a non-zero MG in the $2D$ $SU(2)$ principal chiral model. Following the original idea of Mack and Petkova [1] for $4D$ non-Abelian gauge theories we formulate a sufficient condition for the MG to be non-vanishing. This condition is expressed in terms of the behaviour of the vortex free energy. However, the vortex condensation mechanism presented here does not specify the definition of a vortex. The only

important issue is that over some characteristic length scale spin field configurations which are crucial for the existence of a non-zero MG behave $Z(2)$-like. With this in mind we separate $Z(2)$ and $SO(3)$ degrees of freedom and construct an effective model for the $Z(2)$ degrees of freedom in a $SO(3)$ background. Assuming that $Z(2)$ degrees of freedom alone carry the whole MG we arrive at an effective Ising-like expression for the $SU(2)$ correlation function. We speculate that due to the non-trivial $SO(3)$ background the effective Ising-like coupling might be smaller than the critical Ising coupling for all values of the bare coupling constant. In order to confirm the above assumption we performed numerical simulations and compared the full $SU(2)$ MG with the MG extracted from $Z(2)$ degrees of freedom. We find that $Z(2)$ degrees of freedom reproduce the full $SU(2)$ MG with perfect accuracy. This is observed not only in the standard model but also in the positive link model which due to the complete suppression of links with negative trace is closer to the continuum limit. We summarize that our numerical results as well as the effective model for $Z(2)$ degrees of freedom are in favour of a vortex condensation mechanism.

Finally, we want to stress the following: All arguments presented in this paper being in favour of a vortex theory did not use information on the phase structure of the considered model. Thus, they hold independently of the scenario actually being realized. In particular, they do not depend on whether the conventional scenario (with a non-zero MG at arbitrarily large β and asymptotic freedom) or the scenario advocated in ref. [23] (with a phase transition to a massless phase at a finite value of β) is realized. Moreover, if the conjecture that the $Z(2)$ MG coincides with the full $SU(2)$ MG is correct, then our effective model might help to clarify this important question.

Acknowledgements

We would like to thank M. Faber for many stimulating discussions.

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